Quantum Invariants of Links and New Quantum Field Models

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Abstract

We propose a gauge model of quantum electrodynamics (QED) and its nonabelian generalization from which we derive knot invariants such as the Jones polynomial. Our approach is inspired by the work of Witten who derived knot invariants from quantum field theory based on the Chern-Simon Lagrangian. From our approach we can derive new knot and link invariants which extend the Jones polynomial and give a complete classification of knots and links. From these new knot invariants we have that knots can be completely classified by the power index m of TrR^{-m} where R denotes the R-matrix for braiding and is the monodromy of the Knizhnik-Zamolodchikov equation. A classification table of knots can then be formed where prime knots are classified by prime integer m and nonprime knots are classified by nonprime integer m.

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1 Introduction

In 1989 Witten derived knot invariants such as the Jones polynomial from quantum field theory based on the Chern-Simon Lagrangian [1]. Inspired by Witten's work in this paper we shall derive knot invariants from a gauge model of Quantum electrodynamics (QED) and its nonabelian generalization. From our approach we shall first derive the Jones polynomial and then we derive new knot and link invariants which extend the Jones polynomial and can give a complete classification of knots and links.

This paper is organized as follows. In section 2 we give a brief description of a gauge model of QED and its nonabelian extention. In this paper we shall consider a nonabelian extension with a $SU(2)$ gauge symmetry. With this quantum field model we introduce the partition function and the correlation of a Wilson loop which will be a knot invariant of the trivial knot (also called the unknot). This correlation of Wilson loop will later be generalized to be knot invariants of nontrivial knots and links. To investigate the properties of these partition function and correlations in section 4 we derive a chiral symmetry from the gauge transformation of this new quantum field model. From this chiral symmetry in section 5, section 6 we derive a conformal field theory which contains topics such as the affine Kac-Moody algebra and the Knizhnik-Zamolodchikov equation. This KZ equation is an equation of correlations from which in section 7 we can derive the skein relation of the Jones polynomial. A main point of our theory on the KZ equation is that we can derive two KZ equations which are dual to each other. From these two KZ equations we derive a quantum group structure for the W matrices from which a Wilson loop is formed. Then from the correlation of these W matrices in section 8 and section 9 we derive new knot invariants which extend the Jones polynomial and gives a complete classification of knots. In section 10 we extend the new invariants to the case of links and we compute these new link invariants for some

examples of links. Then in section 11 with the new knot invariants we give a classification table of knots which is formed by using the power index m which comes from these new knot invariants of the form $TrR^{-m}\langle W(z,z)\rangle$ where $W(z,z)$ denotes a Wilson loop and R is the braiding matrix for the quantum group structure and is the monodromy of the two KZ equations.

2 New gauge Model of QED and Nonabelian Extensions

To begin our derivation of knot and link invariants let us first describe a quantum field model. Similar to the Wiener measure for the Brownian motion which is constructed from the integral $\int_{t_0}^{t_1}\left(\frac{dx}{dt}\right)^2 dt$ we construct a measure for QED from the following energy integral:

$$-\frac{1}{2}\int_{s_0}^{s_1}[\frac{1}{2}\left(\frac{dA_1}{ds}-\frac{dA_2}{ds}\right)^2+\sum_{i=1}^{2}\left(\frac{dz}{ds}-ieA_iz\right)^*\left(\frac{dz}{ds}-ieA_iz\right)]ds \tag{1}$$

where s denotes the proper time in relativity; e denotes the electric charge and the complex variable z, real variables A_1, A_2 represent one electron and two photons respectively. By extending ds to $(ih+\beta)ds$ we get a quantum theory of QED where $h>0$ denotes the Planck constant and $\beta>0$ is a constant related to absolute temperature.

The integral (1) has the following gauge symmetry:

$$z'(s)=z(s)e^{iea(s)}, \quad A_i'(s)=A_i(s)+\frac{da}{ds} \quad i=1,2 \tag{2}$$

where $a(s)$ is a real valued function.

We remark that a main feature of (1) is that it is not formulated with the four-dimensional space-time but is formulated with the one dimensional proper time. We refer to [2] for the physical motivation of this new QED theory.

We can generalize the above QED model with $U(1)$ gauge symmetry to QCD type models with nonabelian gauge symmetry. As an illustration let us consider $SU(2)$ gauge symmetry. Similar to (1) we consider the following energy integral:

$$L:=-\frac{1}{2}\int_{s_0}^{s_1}[\frac{1}{2}tr(D_1A_2-D_2A_1)^*(D_1A_2-D_2A_1)+(D_1Z)^*(D_1Z)+(D_2Z)^*(D_2Z)]ds \tag{3}$$

where $Z=(z_1,z_2)^T$ is a two dimensional complex vector; $A_j=\sum_{k=1}^3 A_j^k t^k$ $(j=1,2)$ where A_j^k denotes a real component of a gauge field A^k; t^k denotes a generator of $SU(2)$; and $D_j=\frac{d}{ds}-igA_j$ $(j=1,2)$ where g denotes the charge of interaction. From (3) we can develop a QCD type model as similar to that for the QED model. We have that (3) is invariant under the following gauge transformation:

$$\begin{aligned} Z'(s) &= U(a(s))Z(s) \\ A_j'(s) &= U(a(s))A_j(s)U^{-1}(a(s))+U(a(s))\frac{dU^{-1}(a(s))}{ds}, j=1,2 \end{aligned} \tag{4}$$

where $U(a(s))=e^{-a(s)}$ and $a(s)=\sum_k a^k(s)t^k$. We shall mainly consider the case that a is a function of the form $a(s)=\omega_1(r(s))$ where ω_1 and r are analytic functions.

3 Knot Invariants

Since (3) is not formulated with the space-time, as analogous to the approach of Witten on knot invariants [1] the following partition function will be shown to be a topological invariant for knots:

$$\langle TrW_R\rangle := \int DA_1DA_2DZe^{L}TrW_R(C) \tag{5}$$

where

$$W_R(C) := W(r_0, r_1) := Pe^{\int_C A_i dx^i} \tag{6}$$

which may be called a Wilson loop as analogous to the usual Wilson loop [1] where C denotes a closed curve of the following form

$$C(s) = (x^1(r(s)), x^2(r(s))), s_0 \leq s \leq s_1 \tag{7}$$

where r is an analytic function such that $r_0 := r(s_0) = r(s_1) := r_1$. This closed curve C is in a two dimensional phase plane (x^1, x^2) which is dual to (A_1, A_2). We let this closed curve C represents the projection of a knot in this two dimensional space. As usual the notation P in the definition of $W_R(C)$ denotes a path-ordered product and R denotes a representation of $SU(2)$ [3][4].

We remark that we also extend the definition of $W_R(C)$ to the case that C is not a closed curve with $r_0 \neq r_1$.

We shall show that (5) is a topological invariant for a trivial knot. This means that in (5) the closed curve C represents a trivial knot. We shall extend (5) to let C represent nontrivial knot.

Our aim is to compute the above knot invariant and its generalization to knot invariants of nontrivial knots which will be defined.

4 Chiral Symmetry

For a given curve $C(s) = (x^1(r(s)), x^2(r(s))), s_0 \leq s \leq s_1$ which may not be a closed curve we define $W(r_0, r_1)$ by (6) where $r_0 = r(s_0)$ and $r_1 = r(s_1)$. Then under an analytic gauge transformation we have the following chiral symmetry:

$$W(r_0, r_1) \mapsto U(\omega(r_1))W(r_0, r_1)U^{-1}(\omega(r_0)) \tag{8}$$

where ω denotes an analytic function. This chiral symmetry is analogous to the chiral symmetry of the usual nonabelian guage theory where U denotes an element of $SU(2)$ [3]. We may extend (9) by extending r to complex variable z to have the following chiral symmetry:

$$W(z_0, z_1) \mapsto U(\omega(z_1))W(z_0, z_1)U^{-1}(\omega(z_0)) \tag{9}$$

This analytic continuation corresponds to the complex transformation $s \mapsto (ih + \beta)s$ for describing quantum physics.

From this chiral symmetry we have the following formulas for the variations $\delta_\omega W$ and $\delta_{\omega'} W$ with respect to the chiral symmetry:

$$\delta_\omega W(z, z') = W(z, z')\omega(z) \tag{10}$$

and

$$\delta_{\omega'} W(z, z') = -\omega'(z')W(z, z') \tag{11}$$

where z and z' are independent variables and $\omega'(z') = \omega(z)$ when $z' = z$. In (10) the variation is with respect to the z variable while in (11) the variation is with respect to the z' variable. This two-side-variations is possible when $z \neq z'$.

5 Affine Kac-Moody Algebra

Let us define

$$J(z) := -kW^{-1}(z, z')\partial_z W(z, z') \tag{12}$$

where $k > 0$ is a constant. As analogous to the WZW model [8][6] J is a generator of the chiral symmetry for (10).

Let us consider the following correlation

$$\langle W_R A(z) \rangle := \int DA_1 DA_2 DZ e^L W_R(C) A(z) \tag{13}$$

By taking a gauge transformation on this correlation and by the gauge invariance of (3) we can derive a Ward identity from which we have the following relation:

$$\delta_\omega A(z) = \frac{-1}{2\pi i} \oint_z dw \omega(w) J(w) A(z) \tag{14}$$

where $\delta_\omega A$ denotes the variation of the field A with respect to the chiral symmetry and the closed line integral $\oint$ is with center z and we let the generator J be given by (12). We remark that our approach here is analogous to the WZW model in conformal field theory [8].

From (9) and (12) we have that the variation $\delta_\omega J$ of the generator J of the chiral symmetry is given by [8][6]:

$$\delta_\omega J = [J, \omega] - k\partial_z \omega \tag{15}$$

From (14) and (15) we have that J satisfies the following relation of current algebra [8][6][7]:

$$J^a(w) J^b(z) = \frac{k\delta_{ab}}{(w-z)^2} + \sum_c i f_{abc} \frac{J^c(z)}{(w-z)} \tag{16}$$

where we write

$$J(z) = \sum_a J^a(z) t^a = \sum_a \sum_{n=-\infty}^{\infty} J_n^a z^{-n-1} t^a \tag{17}$$

Then from (16) we can show that J_n^a satisfy the following affine Kac-Moody algebra [8][6][7]:

$$[J_m^a, J_n^b] = i f_{abc} J_{m+n}^c + km\delta_{ab}\delta_{m+n,0} \tag{18}$$

where the constant k is called the central extension or the leval of the Kac-Moody algebra.

Let us consider another generator of the chiral symmetry for (11) given by

$$J'(z') = k\partial_{z'} W(z, z') W^{-1}(z, z') \tag{19}$$

Similar to J by the following correlation:

$$\langle A(z') W_R \rangle := \int DA_1 DA_2 DZ A(z') W_R(C) e^L \tag{20}$$

we have the following formula for J':

$$\delta_{\omega'} A(z') = \frac{-1}{2\pi i} \oint_{z'}^{-} dw A(z') J'(w)(-\omega')(w) = \frac{-1}{2\pi i} \oint_{z'} dw A(z') J'(w) \omega'(w) \tag{21}$$

where $\oint^-$ denotes an integral with clockwise direction while $\oint$ denotes an integral with counterclockwise direction. We remark that this two-side variation from (13) and (20) is important for deriving the two KZ equations which are dual to each other.

Then similar to (15) we also we have

$$\delta_{\omega'} J' = [\omega', J'] - k\partial_{z'} \omega' \tag{22}$$

Then from (21) and (22) we can derive the current algebra and the Kac-Moody algebra for J' which are of the same form of (16) and (18).

6 Dual Knizhnik-Zamolodchikov Equation

Let us first consider (10). From (14) and (10) we have

$$J^a(z)W(w,w') \sim \frac{-t^a W(w,w')}{z-w} \tag{23}$$

Let us define an energy-momentum tensor $T(z)$ by

$$T(z) := \frac{1}{k+g}\sum_a : J^a(z)J^a(z) : \tag{24}$$

where g is the dual Coxter number. In (24) the symbol : ... : denotes normal ordering. This is the Sugawara construction of energy-momentum tensor where the appearing of g is from a renormalization of quantum effect by requiring the operator product expansion of T with itself to be of the following form [8] [6] [7]:

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{25}$$

for some constant $c = \frac{kd}{k+g}$ where d denotes the dimension of $SU(2)$.

Then we have the following TW operator product:

$$T(z)W(w,w') \sim \frac{\Delta}{(z-w)^2} + \frac{1}{(z-w)}L_{-1}W(w,w') \tag{26}$$

where $L_{-1}W(w,w') = \partial_w W(w,w')$ and

$$\Delta = \frac{\sum_a t^a t^a}{2(k+g)} = \frac{N^2-1}{2N(k+N)} \tag{27}$$

where N is for $SU(N)$.

From (24) and (26) we have the following equation [6][7]:

$$L_{-1}W(w,w') = \frac{1}{k+g}J^a_{-1}J^a_0 W(w,w') \tag{28}$$

Then form (23) we have

$$J^a_0 W(w,w') = -t^a W(w,w') \tag{29}$$

By (28) and (29) we have

$$\partial_z W(z,z') = \frac{-1}{k+g}J^a_{-1}t^a W(z,z') \tag{30}$$

From this equation and by the JW operator product (23) we have the following Knizhnik-Zamolodchikov equation [6] [7]:

$$\partial_{z_i}\langle W(z_1,z_1')\cdots W(z_n,z_n')\rangle = -\frac{1}{k+g}\sum_{j\neq i}^{n}\frac{\sum_a t^a\otimes t^a}{z_i-z_j}\langle W(z_1,z_1')\cdots W(z_n,z_n')\rangle \tag{31}$$

We remark that in deriving (31) we have used line integral expression of operators with counterclockwise direction [6][7].

It is interesting and important that we also have another Knizhnik-Zamolodchikov equation which will be called the dual equation of (31). The derivation of this dual equation is dual to the above derivation

in that the line integral for this derivation of dual equation is with clockwise direction in contrast to counterclockwise direction in the above derivation and that the operator products and their corresponing variables are with reverse order to that in the above derivation.

From (11) and (21) we have a WJ' operator product given by

$$W(w,w')J'^{a}(z') \sim \frac{-W(w,w')t^{a}}{w'-z'} \tag{32}$$

Similar to the above derivation of the KZ equation from (32) we can then derive the following Knizhnik-Zamolodchikov equation which is dual to (31):

$$\partial_{z_i'}\langle W(z_1,z_1')\cdots W(z_n,z_n')\rangle = -\frac{1}{k+g}\sum_{j\neq i}^{n}\langle W(z_1,z_1')\cdots W(z_n,z_n')\rangle \frac{\sum_a t^a\otimes t^a}{z_j'-z_i'} \tag{33}$$

7 Skein Relation for Jones Polynomial

Following the idea of Witten [1], if we cut a knot we get two pieces of curves crossing (or not crossing) each other once. This gives two primary fields $W(z_1,z_2)$ and $W(z_3,z_4)$ where $W(z_1,z_2)$ corresponds to a piece of curve with end points parametrized by z_1 and z_2 and $W(z_3,z_4)$ corresponds to the other piece of curve with end points parametrized by z_3 and z_4. Let us write

$$W(z_i,z_j) = W(z_i,z_i')W^{-1}(z_j,z_j') \tag{34}$$

for $i=1,3$ and $j=2,4$ and for some z_k' with $z_1'=z_2'$ and $z_3'=z_4'$. These two pieces of curves then correspond to the following four-point correlation function:

$$G(z_1,z_2,z_3,z_4) := \langle W(z_1',z_1)W^{-1}(z_2',z_2)W(z_3,z_3')W^{-1}(z_4,z_4')\rangle \tag{35}$$

(In the notation $G(z_1,z_2,z_3,z_4)$ we have supressed the z' variables for simplicity). Then we have [6][7]:

$$G(z_1,z_2,z_3,z_4) = [(z_1-z_3)(z_2-z_4)]^{-2\Delta}G(x) \tag{36}$$

where $\Delta = \frac{N^2-1}{2N(N+k)}$ and $x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$ and from the KZ equation $G(x)$ satisfies the following equation:

$$\frac{dG}{dx} = [\frac{1}{x}P + \frac{1}{x-1}Q]G \tag{37}$$

where

$$P = -\frac{1}{N(N+k)}\begin{pmatrix} N^2-1 & N \\ 0 & -1 \end{pmatrix}, \quad Q = -\frac{1}{N(N+k)}\begin{pmatrix} -1 & 0 \\ N & N^2-1 \end{pmatrix} \tag{38}$$

This equation has two independent conformal block solutions forming a vector space of dimension 2. Let ψ be a vector in this space and let B denotes the braid operation. Then following Witten [1] we have

$$a\psi - bB\psi + B^2\psi = 0 \tag{39}$$

where $a = detB$ and $b = TrB$. Then following Witten [1] from (39) we can derive the following skein relation for the Jones polynomial:

$$\frac{1}{t}V_{L_-} - tV_{L_+} = (t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}})V_{L_0} \tag{40}$$

where V_{L_-}, V_{L_+} and V_{L_0} are the Jones polynomials for undercrossing, overcrossing and zero crossing respectively.

8 New Knot Invariants Extending Jones Polynomial

Let us consider again the correlation $G(z_1, z_2, z_3, z_4)$ in (35) which also have the following form:

$$G(z_1, z_2, z_3, z_4) = \langle W(z_1, z_2)W(z_3, z_4)\rangle \tag{41}$$

From it in this section we shall present a method which is different from the above section to derive new knot invariants. These new knot invariants will extend the Jones polynomial and they will be defined by generalizing (5).

We have that G satisfies the KZ equation for the variables z_1, z_3 and satisfies the dual KZ equation for the variables z_2 and z_4. By solving the KZ equation we have that G is of the form

$$e^{t\log(z_1-z_3)}C_1 \tag{42}$$

where $t := \frac{1}{k+g}\sum_a t^a \otimes t^a$ and C_1 denotes a constant matrix which is independent of the variable $z_1 - z_3$.

Similarly by solving the dual KZ equation we have that G is of the form

$$C_2 e^{-t\log(z_4-z_2)} \tag{43}$$

where C_2 denotes a constant matrix which is independent of the variable $z_4 - z_2$.

From (42), (43) and we let $C_1 = Ae^{-t\log(z_4-z_2)}$, $C_2 = e^{t\log(z_1-z_3)}A$ where A is a constant matrix we have that G is given by

$$G(z_1, z_2, z_3, z_4) = e^{t\log(z_1-z_3)}Ae^{-t\log(z_4-z_2)} \tag{44}$$

Now let $z_2 = z_3$. Then as $z_4 \to z_1$ we have

$$TrG(z_1, z_2, z_2, z_1) = Tre^{i2n\pi t}A =: TrR^{2n}A \qquad n = 0, \pm1, \pm2, ... \tag{45}$$

where $R = e^{i\pi t}$ is the monodromy of the the KZ equation [5]. We remark that (45) is a multivalued function. From (45) we have the following relation between the partition function Z and the matrix A:

$$A = IZ \tag{46}$$

where I denotes the identity matrix. Now let C be a closed curve in the complex plane with initial and final end points z_1. Then the following correlation function

$$Tr\langle W(z_1, z_1)\rangle = Tr\langle W(z_1, z_2)W(z_2, z_1)\rangle \tag{47}$$

which is the definition (5) defined along the curve C, with $W(z_1, z_1) = W(z_1, z_2)W(z_2, z_1)$, can be regarded as a knot invariant of the trivial knot in the three dimensional space whose porjection in the complex plane is the curve C. Indeed, from (41) and (45) we can compute (47) which is given by:

$$Tr\langle W(z_1, z_1)\rangle = ZTrR^{2n} \qquad n = 0, \pm1, \pm2, ... \tag{48}$$

From (48) we see that (47) is independent of the closed curve C which represents the projection of a trivial knot and thus can be regarded as a knot invariant for the trivial knot.

In the following let us extend the definition (47) to knot invariants for nontrivial knots.

Since R is the monodromy of the KZ equation, we have a branch cut such that

$$\langle W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4)\rangle = R\langle W(z_1, w)W(w, z_2)W(z_3, w)W(w, z_4)\rangle \tag{49}$$

where z_1 and z_3 denote two points on a closed curve such that along the direction of the curve the point z_1 is before the point z_3. From (49) we have

$$W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4) = RW(z_1, w)W(w, z_2)W(z_2, w)W(w, z_4) \tag{50}$$

Similarly for the dual KZ equation we have

$$\langle W(z_1,w)W(w,z_4)W(z_3,w)W(w,z_2)\rangle = \langle W(z_1,w)W(w,z_2)W(z_3,w)W(w,z_4)\rangle R^{-1} \tag{51}$$

and

$$W(z_1,w)W(w,z_4)W(z_3,w)W(w,z_2) = W(z_1,w)W(w,z_2)W(z_3,w)W(w,z_4)R^{-1} \tag{52}$$

where z_2 before z_4. From (50) and (52) we have

$$W(z_3,z_4)W(z_1,z_2) = RW(z_1,z_2)W(z_3,z_4)R^{-1} \tag{53}$$

where z_1 and z_2 denote the end points of a curve which is before a curve with end points z_3 and z_4. From (53) we see that the algebraic structure of these W matrices is analogous to the quasi-triangular quantum group [5][7]. Now we let $W(z_i,z_j)$ represent a piece of curve with initial end point z_i and final end point z_j. Then we let

$$W(z_1,z_2)W(z_3,z_4) \tag{54}$$

represent two pieces of uncrossing curve. Then by interchanging z_1 and z_3 we have

$$W(z_3,w)W(w,z_2)W(z_1,w)W(w,z_4) \tag{55}$$

represent the curve specified by $W(z_1,z_2)$ upcrossing the curve specified by $W(z_3,z_4)$ at z. Similarly by interchanging z_2 and z_4 we have

$$W(z_1,w)W(w,z_4)W(z_3,w)W(w,z_2) \tag{56}$$

represent the curve specified by $W(z_1,z_2)$ undercrossing the curve specified by $W(z_3,z_4)$ at z.

Now for a closed curve we may cut it into a sum of parts which are formed by two pieces of curve crossing or not crossing each other. Each of these parts is represented by (54), (55) or (56). Then we may define a correlation for a knot whose projection is this closed curve by the following form:

$$Tr\langle\cdots W(z_3,z)W(z,z_2)W(z_1,z)W(z,z_4)\cdots\rangle \tag{57}$$

where we use (55) as an example to represent the state of the two pieces of curve specified by $W(z_1,z_2)$ and $W(z_3,z_4)$. The $\cdots$ means multiplications of a sequence of parts represented by (54), (55) or (56) according to the state of each part. The order of the sequence in (57) follows the order of the parts given by the direction of the knot.

We shall show that (57) is a knot invariant for a given knot. In the following let us consider some examples to illustrate the way to define (57) and to show that (57) is a knot invariant. We shall also derive the three Reidemeister moves for the equivalence of knots.

Let us first consider a knot in Fig. 1. For this knot we have that (57) is given by

$$Tr\langle W(z_2,w)W(w,z_2)W(z_1,w)W(w,z_1)\rangle \tag{58}$$

where the product of W is from the definition (55). In applying (55) we let z_1 as the starting and the final point. We remark that the W matrices must be put together to follow the definition (55) and they are not separated to follow the direction of the knot.

Then we have that (58) is equal to

$$\begin{aligned} & Tr\langle W(w,z_2)W(z_1,w)W(w,z_1)W(z_2,w)\rangle \\ = \; & Tr\langle RW(z_1,w)W(w,z_2)R^{-1}RW(z_2,w)W(w,z_1)R^{-1}\rangle \\ = \; & Tr\langle W(z_1,z_2)W(z_2,z_1)\rangle \\ = \; & Tr\langle W(z_1,z_1)\rangle \end{aligned} \tag{59}$$

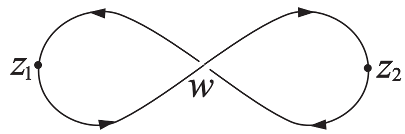

Fig.1

where we have used (53). We see that (59) is just the knot invariant (47) of a trivial knot. Thus the knot in Fig.1 is with the same knot invariant of a trivial knot and this agrees with the fact that this knot is topologically equivalent to a trivial knot.

Then let us derive the Reidemeister move 1. Consider the diagram in Fig.2. We have that by (55) the definition (57) for this diagram is given by:

$$\begin{aligned}
& Tr\langle W(z_2,w)W(w,z_2)W(z_1,w)W(w,z_3)\rangle \\
= \ & Tr\langle W(z_2,w)RW(z_1,w)W(w,z_2)R^{-1}W(w,z_3)\rangle \\
= \ & Tr\langle W(z_2,w)RW(z_1,z_2)R^{-1}W(w,z_3)\rangle \\
= \ & Tr\langle R^{-1}W(w,z_3)W(z_2,w)RW(z_1,z_2)\rangle \\
= \ & Tr\langle W(z_2,w)W(w,z_3)W(z_1,z_2)\rangle \\
= \ & Tr\langle W(z_2,z_3)W(z_1,z_2)\rangle \\
= \ & Tr\langle W(z_1,z_3)\rangle
\end{aligned} \tag{60}$$

where$W(z_1,z_3)$ represent a piece of curve with initial end point z_1 and final end point z_3 which has no crossing. When Fig.2 is a part of a knot we can also derive a result similar to (60) which is for the Reidemeister move 1. This shows that the Reidemeister move 1 holds.

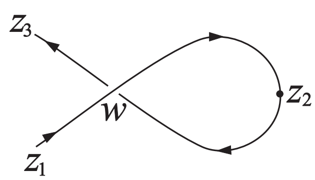

Fig.2

Then let us derive Reidemeister move 2. By (55) we have that the definition (57) for the two pieces of curve in Fig.3a is given by

$$Tr\langle W(z_5,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_6)\cdot W(z_4,w_2)W(w_2,z_3)W(z_2,w_2)W(w_2,z_5)\rangle \tag{61}$$

where the two products of W separated by the $\cdot$ are for the two crossings in Fig.3a. We have that (61)

is equal to

$$
\begin{aligned}
& Tr\langle W(z_4,w_2)W(w_2,z_3)W(z_2,w_2)W(w_2,z_5)\cdot W(z_5,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_4,w_2)W(w_2,z_3)W(z_2,w_2)W(w_2,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_4,w_2)W(w_2,z_3)W(z_2,w_2)RW(w_1,z_2)W(w_2,w_1)R^{-1}W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_4,w_2)RW(z_2,w_2)W(w_2,z_3)W(w_1,z_2)W(w_2,w_1)R^{-1}W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_4,w_2)RW(z_2,z_3)W(w_1,z_2)W(w_2,w_1)R^{-1}W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_4,w_2)W(w_1,z_2)W(z_2,z_3)RW(w_2,w_1)R^{-1}W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_4,w_2)W(w_1,z_3)RW(w_2,w_1)R^{-1}W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle RW(w_1,z_3)W(z_4,w_2)W(w_2,w_1)R^{-1}W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle RW(w_1,z_3)W(z_4,w_1)R^{-1}W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_4,w_1)W(w_1,z_3)W(z_1,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_4,w_1)RW(z_1,w_1)W(w_1,z_3)R^{-1}W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_1,z_3)R^{-1}W(w_1,z_6)W(z_4,w_1)R\rangle \\
= \ & Tr\langle W(z_1,z_3)W(z_4,w_1)W(w_1,z_6)\rangle \\
= \ & Tr\langle W(z_1,z_3)W(z_4,z_6)\rangle
\end{aligned} \tag{62}
$$

where we have used (53). This shows that the diagram in Fig.3a is equivalent to two uncrossing curves. When Fig.3a is a part of a knot we can also derive a result similar to (62) for the Reidemeister move 2. This shows that the Reidemeister move 2 holds. As an illustration let us consider the knot in Fig. 3b

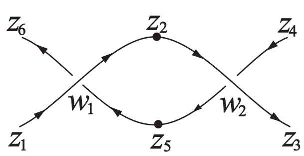

Fig.3a

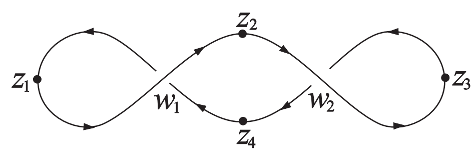

Fig.3b

which is related to the Reidemeister move 2. By (55) we have that the definition (57) for this knot is given by

$$
\begin{aligned}
& Tr\langle W(z_3,w_2)W(w_2,z_3)W(z_2,w_2)W(w_2,z_4)\cdot W(z_4,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_1)\rangle \\
= \ & Tr\langle RW(z_2,w_2)W(w_2,z_3)W(z_3,w_2)W(w_2,z_4)\cdot W(z_4,w_1)W(w_1,z_1)W(z_1,w_1)W(w_1,z_2)R^{-1}\rangle \\
= \ & Tr\langle W(z_2,w_2)W(w_2,z_3)W(z_3,w_2)W(w_2,z_4)\cdot W(z_4,w_1)W(w_1,z_1)W(z_1,w_1)W(w_1,z_2)\rangle \\
= \ & Tr\langle W(z_2,z_2)\rangle
\end{aligned} \tag{63}
$$

where we let the curve be with z_2 as the initial and final end point and we have used (50) and (52). This shows that the knot in Fig.3b is with the same knot invariant of a trivial knot. This agrees with the fact that this knot is equivalent to the trivial knot.

Let us then consider a trefoil knot in Fig.4a. By (55) and similar to the above examples we have that

the definition (57) for this knot is given by:

$$
\begin{aligned}
& Tr\langle W(z_4,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_5)\cdot W(z_2,w_2)W(w_2,z_6) \\
& W(z_5,w_2)W(w_2,z_3)\cdot W(z_6,w_3)W(w_3,z_4)W(z_3,w_3)W(w_3,z_1)\rangle \\
= \; & Tr\langle W(z_4,w_1)RW(z_1,w_1)W(w_1,z_2)R^{-1}W(w_1,z_5)\cdot W(z_2,w_2)RW(z_5,w_2) \\
& W(w_2,z_6)R^{-1}W(w_2,z_3)\cdot W(z_6,w_3)RW(z_3,w_3)W(w_3,z_4)R^{-1}W(w_3,z_1)\rangle \\
= \; & Tr\langle W(z_4,w_1)RW(z_1,z_2)R^{-1}W(w_1,z_5)\cdot W(z_2,w_2)RW(z_5,z_6)R^{-1}W(w_2,z_3)\cdot \\
& W(z_6,w_3)RW(z_3,z_4)R^{-1}W(w_3,z_1)\rangle \\
= \; & Tr\langle W(z_4,w_1)RW(z_1,z_2)W(z_2,w_2)W(w_1,z_5)W(z_5,z_6)R^{-1}W(w_2,z_3)\cdot \\
& W(z_6,w_3)RW(z_3,z_4)R^{-1}W(w_3,z_1)\rangle \\
= \; & Tr\langle W(z_4,w_1)RW(z_1,w_2)W(w_1,z_6)R^{-1}W(w_2,z_3) \\
& W(z_6,w_3)RW(z_3,z_4)R^{-1}W(w_3,z_1)\rangle \\
= \; & Tr\langle W(z_4,w_1)W(w_1,z_6)W(z_1,w_2)W(w_2,z_3) \\
& W(z_6,w_3)RW(z_3,z_4)R^{-1}W(w_3,z_1)\rangle \\
= \; & Tr\langle W(z_4,z_6)W(z_1,z_3)W(z_6,w_3)RW(z_3,z_4)R^{-1}W(w_3,z_1)\rangle \\
= \; & Tr\langle R^{-1}W(w_3,z_1)W(z_4,z_6)W(z_1,z_3)W(z_6,w_3)RW(z_3,z_4)\rangle \\
= \; & Tr\langle W(z_4,z_6)W(w_3,z_1)R^{-1}W(z_1,z_3)W(z_6,w_3)RW(z_3,z_4)\rangle \\
= \; & Tr\langle RW(z_3,z_6)W(w_3,z_1)R^{-1}W(z_1,z_3)W(z_6,w_3)\rangle \\
= \; & Tr\langle W(w_3,z_1)W(z_3,z_6)W(z_1,z_3)W(z_6,w_3)\rangle \\
= \; & Tr\langle W(z_6,z_1)W(z_3,z_6)W(z_1,z_3)\rangle
\end{aligned} \tag{64}
$$

where we have repeatly used (53). Then we have that (64) is equal to:

$$
\begin{aligned}
& Tr\langle W(z_6,w_3)W(w_3,z_1)W(z_3,w_3)W(w_3,z_6)W(z_1,z_3)\rangle \\
= \; & Tr\langle RW(z_3,w_3)W(w_3,z_1)W(z_6,w_3)W(w_3,z_6)W(z_1,z_3)\rangle \\
= \; & Tr\langle RW(z_3,w_3)RW(z_6,w_3)W(w_3,z_1)R^{-1}W(w_3,z_6)W(z_1,z_3)\rangle \\
= \; & Tr\langle W(z_3,w_3)RW(z_6,z_1)R^{-1}W(w_3,z_6)W(z_1,z_3)R\rangle \\
= \; & Tr\langle W(z_3,w_3)RW(z_6,z_3)W(w_3,z_6)\rangle \\
= \; & Tr\langle W(w_3,z_6)W(z_3,w_3)RW(z_6,z_3)\rangle \\
= \; & Tr\langle RW(z_3,w_3)W(w_3,z_6)W(z_6,z_3)\rangle \\
= \; & Tr\langle RW(z_3,z_3)\rangle
\end{aligned} \tag{65}
$$

where we have used (50) and (53). We see that (65) is a knot invariant for the trefoil knot in Fig.4a.

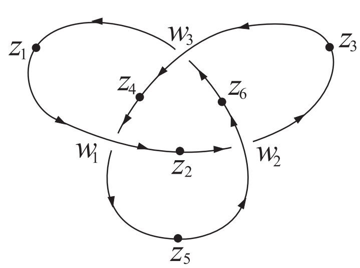

Fig.4a

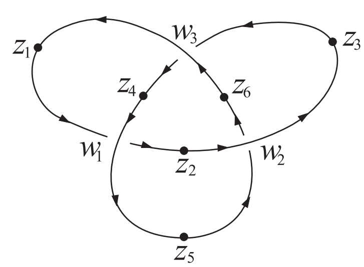

Fig.4b

Then let us consider the trefoil knot in Fig. 4b which is the mirror image of the trefoil knot in Fig.4a. The definition (57) for this knot is given by:

$$
\begin{aligned}
& Tr\langle W(z_1,w_1)W(w_1,z_5)W(z_4,w_1)W(w_1,z_2)\cdot W(z_5,w_2)W(w_2,z_3)W(z_2,w_2)W(w_2,z_6)\cdot \\
& W(z_3,w_3)W(w_3,z_1)W(z_6,w_3)W(w_3,z_4)\rangle \\
= \; & Tr\langle W(z_5,z_1)W(z_2,z_5)W(z_1,z_2)\rangle
\end{aligned} \tag{66}
$$

where similar to (64) we have repeatly used (53). Then we have that (66) is equal to:

$$
\begin{aligned}
& Tr\langle W(z_5,z_1)W(z_2,w_1)W(w_1,z_5)W(z_1,w_1)W(w_1,z_2)\rangle \\
= & Tr\langle W(z_5,z_1)W(z_2,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_5)R^{-1}\rangle \\
= & Tr\langle W(z_5,z_1)W(z_2,w_1)RW(z_1,w_1)W(w_1,z_2)R^{-1}W(w_1,z_5)R^{-1}\rangle \\
= & Tr\langle R^{-1}W(z_5,z_1)W(z_2,w_1)RW(z_1,z_2)R^{-1}W(w_1,z_5)\rangle \\
= & Tr\langle W(z_2,w_1)W(z_5,z_2)R^{-1}W(w_1,z_5)\rangle \\
= & Tr\langle W(z_5,z_2)R^{-1}W(w_1,z_5)W(z_2,w_1)\rangle \\
= & Tr\langle W(z_5,z_2)W(z_2,w_1)W(w_1,z_5)R^{-1}\rangle \\
= & Tr\langle W(z_5,z_5)R^{-1}\rangle
\end{aligned}
\tag{67}
$$

where we have used (52) and (53). We see that this is a knot invariant for the trefoil knot in Fig.4b. we notice that the knot invariants for the two trefoil knots are different. This shows that these two trefoil knots are not topologically equivalent.

Then let us derive the Reidemeister move 3. We have that the definition (57) for the diagram in Fig.5a is given by

$$
\begin{aligned}
& Tr\langle W(z_7,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_8)\cdot W(z_8,w_3)W(w_3,z_6)W(z_5,w_3)W(w_3,z_9)\cdot \\
& W(z_4,w_2)W(w_2,z_3)W(z_2,w_2)W(w_2,z_5)\rangle
\end{aligned}
\tag{68}
$$

where we let the the curve with end points z_7, z_9 starts first, then the curve with end points z_1, z_3 starts second. Similar to the derivation of the above invariants by (50), (52), (53) we have that (68) is equivalent to the following correlation:

$$
Tr\langle W(z_7,z_9)W(z_1,z_3)W(z_5,z_6)R^{-1}W(z_4,z_5)\rangle \tag{69}
$$

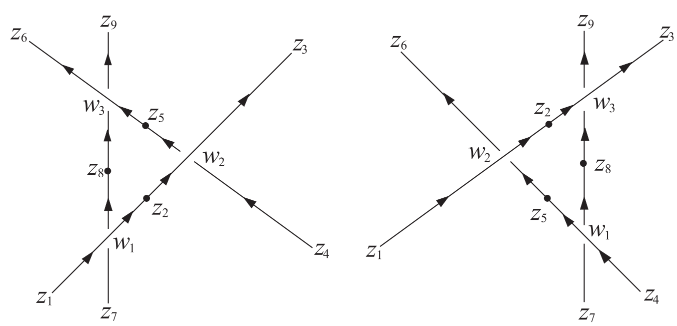

Fig.5a Fig.5b

On the other hand the definition (57) for the diagram Fig.5b is given by

$$
\begin{aligned}
& Tr\langle W(z_7,w_1)W(w_1,z_5)W(z_4,w_1)W(w_1,z_8)\cdot W(z_8,w_3)W(w_3,z_3)W(z_2,w_3)W(w_3,z_9)\cdot \\
& W(z_5,w_2)W(w_2,z_2)W(z_1,w_2)W(w_2,z_6)\rangle
\end{aligned}
\tag{70}
$$

where the ordering of the three curves is the same as that in Fig.5a. By (50), (52), (53) we have that (70) is equal to

$$
Tr\langle W(z_7,z_9)W(z_4,z_5)W(z_1,z_3)R^{-1}W(z_5,z_6)\rangle \tag{71}
$$

Then by reversing the ordering of the curves with end points z_1, z_3 and with end points z_4, z_6 respectively we have that both (69) and (70) are equal to

$$
Tr\langle W(z_7,z_9)W(z_4,z_6)W(z_1,z_3)R^{-1}\rangle \tag{72}
$$

This shows that Fig.5a is equivalent to Fig.5b and this gives the Reidemeister move 3.

From the above examples we see that knots can be classified by the number m of product of R and R^{-1} matrices. More calculations and examples of the above knot invariants will be given elsewhere.

9 Classification of Knots and Links

With the knot invariants in the above section we can now give a classification of knots. Let K_1 and K_2 be two knots. Since the two W-products $W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4)$ and $W(z_1, z_2)W(z_3, z_4)$ faithfully represent two oriented pieces of curves which are crossing or not crossing to each other we have that from the orientation of a knot K we have that the product of sequence of W-matrices (We may call this product as a generalized Wilson loop) which are formed according to the orientation of K in the correlation (57) of defining an invariant of the knot K faithfully represents the knot K. From this we have that K_1 and K_2 are topological equivalent if and only if their generalized Wilson loops can be transformed to each other by using (50), (52) and (53). We note that in the above section we can derive the Reidemeister moves by using (50), (52) and (53). Thus this equivalence of K_1 and K_2 represented by their generalized Wilson loops agrees with the fact that K_1 and K_2 are topologically equivalent if and only if they can be transformed to each other by the Reidemeister moves.

Then since each knot can be changed to a trivial knot by applying braiding operation [9] which is equivalent to (50), (52) and (53) it follows that these new knot invariants can be equivalently transformed to the form $TrR^{-m}\langle W(z_1, z_1)\rangle$ where m is an integer and $Tr\langle W(z_1, z_1)\rangle$ is the knot invariant for the trivial knot. This form can also be shown by a direct computation which is similar to the computation of the invariant of the trivial knot. We notice that since this new knot invariant is of the form $Tr\langle R^{-m}W(z, z)\rangle$ we have that knots can be completely classified by the power index m of R.

Similar to the case of knots we have that the generalized Wilson loop for a link can faithfully represents this link and that the correlation (57) of the generalized Wilson loop of this link is an invariant which can completely classifies links. For the case of link as similar to the case of knot the ordering of the crossings $W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4)$ can be given by following the orientation of each component of a link. When a component of a link has been traced for one loop the crossings $W(z_3, w)W(w, z_2)W(z_1, w)W(w, z_4)$ related to this component can be ordered and the pieces $W(w_i, z_j)$ related to these crossings (which may come from other components of the link) have also been required being ordered. In the following section we give some examples to illustrate the formation and computations of this new link invariant.

10 Examples of New Link Invariants

Let us first consider the link in Fig.6a. We may let the two knots of this link be with z_1 and z_4 as the initial and final end point respectively. We let the ordering of these two knots be such that when the z parameter goes one loop on one knot then the z parameter for another knot also goes one loop. The correlation (57) for this link is given by:

$$Tr\langle W(z_3, w_1)W(w_1, z_2)W(z_1, w_1)W(w_1, z_4) \cdot W(z_4, w_2)W(w_2, z_1)W(z_2, w_2)W(w_2, z_3)\rangle \tag{73}$$

We let the ordering of the W-matrices in (73) be such that $W(z_1, z_2)$ and $W(z_4, z_3)$ start first. Then next $W(z_2, z_1)$ and $W(z_3, z_4)$ follows. Form this ordering we have that (73) is equal to:

$$\begin{aligned} & Tr\langle RW(z_1, w_1)W(w_1, z_2)W(z_3, w_1)W(w_1, z_4) \cdot W(z_4, w_2)W(w_2, z_3)W(z_2, w_2)W(w_2, z_1)R^{-1}\rangle \\ = \; & Tr\langle W(z_1, z_2)W(z_3, z_4)W(z_4, z_3)W(z_2, z_1)\rangle \\ = \; & Tr\langle W(z_2, z_2)W(z_3, z_3)\rangle \end{aligned} \tag{74}$$

where we have used (50) and (52). Since by definition (57) we have that $Tr\langle W(z_2, z_2)W(z_3, z_3)\rangle$ is the knot invariant for two unlinking trivial knots, equation (74) shows that the link in Fig.6a is topologically equivalent to two unlinking trivial knots. Similarly we can show that the link in Fig.6b is topologically equivalent to two unlinking trivial knots.

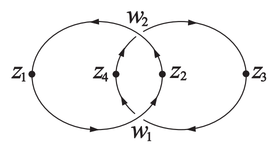

Fig.6a

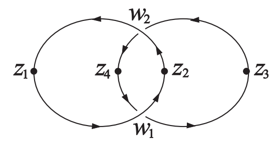

Fig.6b

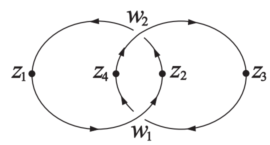

Fig.7a

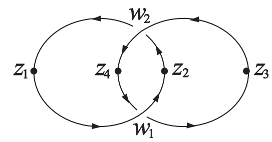

Fig.7b

Let us then consider the Hopf link in Fig.7a. The correlation (57) for this link is given by:

$$Tr\langle W(z_3,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_4)\cdot W(z_2,w_2)W(w_2,z_3)W(z_4,w_2)W(w_2,z_1)\rangle \tag{75}$$

The ordering of the W-matrices in (75) is such that $W(z_1,z_2)$ starts first and $W(z_3,z_4)$ follows it. Then next we let $W(z_2,z_1)$ starts first and $W(z_4,z_3)$ follows it. The ordering is such that when the z parameter goes one loop in one knot of the link we have that the z parameter also goes one loop on the other knot. From the ordering we have that (75) is equal to:

$$\begin{aligned} & Tr\langle RW(z_1,w_1)W(w_1,z_2)W(z_3,w_1)W(w_1,z_4)\cdot \\ & W(z_2,w_2)W(w_2,z_1)W(z_4,w_2)W(w_2,z_3)R^{-1}\rangle \\ = \; & Tr\langle W(z_1,z_2)W(z_3,z_4)W(z_2,z_1)W(z_4,z_3)\rangle \end{aligned} \tag{76}$$

Then let us consider the following correlation:

$$Tr\langle R^{-2}W(z_3,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_4)\cdot W(z_2,w_2)W(w_2,z_3)W(z_4,w_2)W(w_2,z_1)\rangle \tag{77}$$

We let the ordering of the W-matrices in (77) be such that $W(z_1,z_2)$ starts first and $W(z_4,z_3)$ follows it. Then next $W(z_2,z_1)$ starts first and $W(z_3,z_4)$ follows it. From the ordering we have that (77) is equal to:

$$\begin{aligned} & Tr\langle R^{-2}RW(z_1,w_1)W(w_1,z_2)W(z_3,w_1)W(w_1,z_4)\cdot W(z_2,w_2)W(w_2,z_1)W(z_4,w_2)W(w_2,z_3)R\rangle \\ = \; & Tr\langle W(z_1,z_2)W(z_3,z_4)W(z_2,z_1)W(z_4,z_3)\rangle \end{aligned} \tag{78}$$

On the other hand from the ordering of (77) we have that (77) is equal to:

$$\begin{aligned} & Tr\langle R^{-2}W(z_3,w_1)RW(z_1,w_1)W(w_1,z_2)R^{-1} \\ & W(w_1,z_4)W(z_2,w_2)RW(z_4,w_2)W(w_2,z_3)R^{-1}W(w_2,z_1)\rangle \\ = \; & Tr\langle R^{-2}W(z_3,w_1)RW(z_1,z_2)R^{-1}W(w_1,z_4)W(z_2,w_2)RW(z_4,z_3)R^{-1}W(w_2,z_1)\rangle \\ = \; & Tr\langle R^{-2}W(z_3,w_1)RW(z_1,z_2)W(z_2,w_2)W(w_1,z_4)W(z_4,z_3)R^{-1}W(w_2,z_1)\rangle \\ = \; & Tr\langle R^{-2}W(z_3,w_1)RW(z_1,w_2)W(w_1,z_3)R^{-1}W(w_2,z_1)\rangle \\ = \; & Tr\langle R^{-2}W(z_3,w_1)W(w_1,z_3)W(z_1,w_2)W(w_2,z_1)\rangle \\ = \; & Tr\langle R^{-2}W(z_3,z_3)W(z_1,z_1)\rangle \end{aligned} \tag{79}$$

where we have repeatly used (53). From (76), (78) and (79) we have that the knot invariant for the Hopf link in Fig.7a is given by:

$$Tr\langle R^{-2}W(z_3,z_3)W(z_1,z_1)\rangle \tag{80}$$

Then let us consider the Hopf link in Fig.7b. The correlation for this link is given by

$$Tr\langle W(z_4,w_1)W(w_1,z_2)W(z_1,w_1)W(w_1,z_3)\cdot W(z_2,w_2)W(w_2,z_4)W(z_3,w_2)W(w_2,z_1)\rangle \tag{81}$$

By a derivation which is dual to the above derivation for the Hopf link in Fig.7a we have that (81) is equal to

$$Tr\langle R^2 W(z_4,z_4)W(z_1,z_1)\rangle \tag{82}$$

We see that the invariants for the above two Hopf links are different. This agrees with the fact that these two links are not topologically equivalent.

We can extend the above computations to other links. As examples let us consider the linking of two trivial knot with linking number n. Similar to the above computations we have that this link which analogous to the Hopf link in Fig.7a is with an invariant equals to $Tr\langle R^{-2n}W(z_4,z_4)W(z_1,z_1)\rangle$. Also for this link which analogous to to the Hopf link in Fig.7b is with an invariant equals to $Tr\langle R^{2n}W(z_4,z_4)W(z_1,z_1)\rangle$.

More calculations and examples of these new link invariants will be given elsewhere.

11 A Classification Table of Knots

In this section let us give some arguments to determine the new knot invariants of prime knots and nonprime knots. We have shown that this new invariant of each knot is of the form $Tr\langle R^{-m}W(z_1,z_1)\rangle$ where m is an integer. This power index m can be regarded as a measure of the complexity of a knot. Let us determine m for prime and nonprime knots. We need only to determine m for knots with positive m since the corresponding mirror image will have negative m if the mirror image is not equivalent to the corresponding knot. We have shown that the invariants for links of two trivial knots with linking number n as in Fig.7a are with the product of R of the form R^{-2n} with $m=2n$. This m is an even number. Now if we insert these links into the knot table of prime knots we see that the prime knots must with m being an odd number (We may refer to the knot table in [10]. We may have nonprime knots which are not in this knot table having the same m as these links. In this case our new link invariants still can distinguish them because the corresponding link invariants are of two-loop form while the corresponding knot invariants are of one-loop form).

Then we expect that for prime knots m is an odd prime number. Computations show that for the knot $\mathbf{3_1}$ we have $m=1$, for the knot $\mathbf{4_1}$ we have $m=3$. Then from some arguments on the effect of R we should have that for the knot $\mathbf{5_1}$ we have $m=5$ and for the knot $\mathbf{5_2}$ we have $m=7$. Then how about the knot $\mathbf{6_1}$? We have that the numbers from 1 to 8 are ocupied by knots and links of two trivial knots. Then 10 is ocupied by the link of two trivial knots with linking number 5. Thus for the knot $\mathbf{6_1}$ we should have $m=9$ or $m=11$. From some argument on the effect of R we should have $m=11$ for $\mathbf{6_1}$. Then is there a knot with $m=9$?

Let us first consider the granny knot (or the square knot) which is a nonprime knot composed with the knot $\mathbf{3_1}$ and its mirror image. This square knot has 6 crossings and 4 alternating crossings and thus its complexity which is measured by the power index m of R is less than that of $\mathbf{5_1}$ which is with 5 alternating crossings. Thus this granny knot is with $m=4$ ($m=3$ has been occupied by $\mathbf{4_1}$). Let us denote this granny knot by $\mathbf{3_1}\star\mathbf{3_1}$ where $\star$ denotes the connected sum of two knots such that the resulting total number of alternating crossings is equal to the total number of alternating crossings of the two knots minus 2.

Then let us consider the reef knot which is a nonprime knot composed with two identical knots $\mathbf{3_1}$. This knot has 6 alternating crossings which is equal to the total number of crossings as that of $\mathbf{6_1}$. Since this knot is nonprime its complexity is less than that of $\mathbf{6_1}$ where the complexity may be measured by the power index m of R. Thus this reef knot is with power index m less than 11. Then if we also regard the total number of alternating crossings of a knot as a way to measure the complexity of a knot we have

that the power index m of this granny knot is greater than 5 since $\mathbf{5_1}$ is with 5 alternating crossings and with $m = 5$. Let us denote this reef knot by $\mathbf{3_1} \times \mathbf{3_1}$ where $\times$ denotes the connected sum for two knots such that the resulting total number of alternating crossings is equal to the total number of alternating crossings of the two knots.

Now let us look for knots with 5 or 6 alternating crossings. Let us consider the nonprime knot $\mathbf{3_1} \star \mathbf{4_1}$ composed with a knot $\mathbf{3_1}$ and a knot $\mathbf{4_1}$ with 7 crossings and 5 alternating crossings. In this case we have that the power index m of $\mathbf{3_1} \star \mathbf{4_1}$ should be greater than that of $\mathbf{5_1}$ which is exactly with 5 alternating crossings since $\mathbf{3_1} \star \mathbf{4_1}$ in addtion has 7 crossings. Then the power index m of $\mathbf{3_1} \star \mathbf{4_1}$ should be less than that of $\mathbf{5_2}$ which is also with 5 alternating crossings but these crossings are arranged in a more complicated way which is an effect of R^2 such that $\mathbf{5_2}$ is with $m = 7$. Thus $\mathbf{3_1} \star \mathbf{4_1}$ is with $m = 6$.

Then we consider the nonprime knot $\mathbf{3_1} \star (\mathbf{3_1} \star \mathbf{3_1})$ with 9 crossings and 5 alternating crossings. The power index of this knot should be less than that of $\mathbf{3_1} \times \mathbf{3_1}$ since it is with 6 alternating crossings. Then the power index of $\mathbf{3_1} \times \mathbf{3_1}$ should be greater than that of $\mathbf{5_2}$ since it has in addition 9 crossings which would be enough for a greater power index m. Then we have that $\mathbf{3_1} \star (\mathbf{3_1} \star \mathbf{3_1})$ is with $m = 8$ since it has 9 crossings and 5 alternating crossings and thus is with the same complexity as $\mathbf{5_2}$. Then since $\mathbf{5_2}$ can not have $m = 8$ we thus have that $\mathbf{3_1} \star (\mathbf{3_1} \star \mathbf{3_1})$ is with $m = 8$.

Then we consider the nonprime knot $\mathbf{3_1} \star \mathbf{5_1}$ with 8 crossings and 6 alternating crossings. The power index m of this knot should be greater than that of $\mathbf{3_1} \times \mathbf{3_1}$ which has exactly 6 alternating crossings. Then the power index m of this knot should be less than that of $\mathbf{6_1}$ which also has 6 alternating crossings but these crossings are arranged in a more complicated way with an effect of R^4 from that of 5_2. Thus $\mathbf{3_1} \star \mathbf{5_1}$ is with $m = 10$ and finally we have that $\mathbf{3_1} \times \mathbf{3_1}$ is with power index $m = 9$.

Thus for m from 1 to 11 we have fill in a suitable knot with power index m (except the case $m = 2$ which is filled in with the Hopf link) such that odd prime numbers are filled with prime knots. In a similar way we may determine the power index m of other prime and nonprime knots. We list the results up to $m = 2^5$ in a form of table.

Type of Knot	Power Index m	Type of Knot	Power Index m
$\mathbf{3_1}$	1	$\mathbf{6_3}$	17
Hopf link	2	$\mathbf{3_1} \times \mathbf{4_1}$	18
$\mathbf{4_1}$	3	$\mathbf{7_1}$	19
$\mathbf{3_1} \star \mathbf{3_1}$	4	$\mathbf{4_1} \star \mathbf{5_1}$	20
$\mathbf{5_1}$	5	$\mathbf{4_1} \star (\mathbf{3_1} \star \mathbf{4_1})$	21
$\mathbf{3_1} \star \mathbf{4_1}$	6	$\mathbf{4_1} \star \mathbf{5_2}$	22
$\mathbf{5_2}$	7	$\mathbf{7_2}$	23
$\mathbf{3_1} \star \mathbf{3_1} \star \mathbf{3_1}$	8	$\mathbf{3_1} \star (\mathbf{3_1} \times \mathbf{3_1})$	24
$\mathbf{3_1} \times \mathbf{3_1}$	9	$\mathbf{3_1} \star (\mathbf{3_1} \star \mathbf{5_1})$	25
$\mathbf{3_1} \star \mathbf{5_1}$	10	$\mathbf{3_1} \star \mathbf{6_1}$	26
$\mathbf{6_1}$	11	$\mathbf{3_1} \star (\mathbf{3_1} \star \mathbf{5_2})$	27
$\mathbf{3_1} \star \mathbf{5_2}$	12	$\mathbf{3_1} \star \mathbf{6_2}$	28
$\mathbf{6_2}$	13	$\mathbf{7_3}$	29
$\mathbf{4_1} \star \mathbf{4_1}$	14	$\mathbf{3_1} \star (\mathbf{3_1} \star \mathbf{3_1}) \star \mathbf{4_1}$	30
$\mathbf{4_1} \star (\mathbf{3_1} \star \mathbf{3_1})$	15	$\mathbf{7_4}$	31
$(\mathbf{3_1} \star \mathbf{3_1}) \star (\mathbf{3_1} \star \mathbf{3_1})$	16	$\mathbf{3_1} \star (\mathbf{3_1} \star \mathbf{3_1}) \star (\mathbf{3_1} \star \mathbf{3_1})$	32

From this table we see that comparable nonprime knots (in a sense from the table) are grouped in each of the intervals between two prime numbers. It is interesting that in each interval nonprime numbers are one-to-one filled with the comparable nonprime knots while prime numbers are filled with prime knots. This grouping property of classification reflects that the new knot invariants (and hence the power index m) give a classification of knots.

Let us find out some rules for the whole classification table. We have shown that even numbers can not be filled with prime knots. Thus even numbers (except 2) can only be filled with nonprime knots. On the other hand each odd nonprime number is between two even numbers which are power indexes of nonprime knots. Thus the knot corresponding to this odd number is in the same group of these two nonprime knots and is comparable with these two nonprime knots and thus must also be a nonprime knot. From this we have that nonprime numbers are power indexes of nonprime knot. Similarly by this grouping property we have that odd prime numbers are power indexes of prime knots.

It is interesting to note that from the above knot table We see that in the $\star$ product the knot $\mathbf{3_1}$ plays the role of the number 2 in the usual multiplication of numbers. Thus the $\star$ product (or the connected sum) is a kind of multiplication corresponding to the usual multiplication of numbers. However the general rule for this multiplication is rather complicated. This reflects the fact that the numbers m are the power indexes of R which are with simple rule for the addition (and not for multipication). From this generalized multiplication (which is the connected sum) of knots which corresponds to the usual multiplication of the power indexes m we also have that prime knots are with odd prime numbers as power indexes and nonprime knots are with nonprime numbers as power indexes.

Thus we have a classification table of knots such that each prime knot corresponds to a prime power index m and each nonprime knot corresponds to a nonprime power index m. More computations to verify this knot table shall be given alsewhere.

12 Conclusion

In this paper from a new quantum field model we have derived a conformal field theory and a quantum group structure for generalized Wilson loops from which we can derive the Jones polynomial and new knot and link invariants which extend the Jones polynomial. We show that these new invariants can completely classify knots and links. These new invariants are in terms of the monodromy R of two Knizhnik-Zamolochikov equations which are dual to each other. In the case of knots these new invariants can be written in the form $Tr\langle R^{-m}W(z,z)\rangle$ from which we may classify knots with the power index m of R. A classification table of knots can then be formed where prime knots are classified with odd prime numbers m and nonprime knots are classified with nonprime numbers m.

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